2.3 Functional Calculus

2.3.1 Functional calculus for bounded operators

In this subsection, we suppose that $A : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator on Hilbert space $\mathcal{H}$. Note that all results in this subsection hold for Banach spaces. Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators. In expressions like $\lambda \,\mathrm{Id} - A$, we usually write $\lambda - A$, omitting the symbol Id for the simplicity.

Definition 2.3.1. The **resolvent set** $\rho(A)$ is defined by

$$\rho(A) := \{\lambda \in \mathbb{C} : \lambda - A \text{ is invertible}\}. \tag{2.3.1}$$

If $\lambda \in \rho(A)$, $(\lambda - A)^{-1}$ is called the **resolvent** of A. The spectrum is

$$\sigma(A) = \mathbb{C} \backslash \rho(A). \tag{2.3.2}$$

Proposition 2.3.2. *The spectrum $\sigma(A)$ is a bounded closed subset of $\mathbb{C}$.*

Proof. If $r > \|A\|$, then $r - A = r(1 - A/r)$ and $\|A/r\| < 1$. Since $\sum_{k=1}^{\infty} \|A/r\|^k < +\infty$, then $1 + \sum_{k=1}^{\infty}(A/r)^k \in \mathcal{H}$ and $r^{-1}(1 + \sum_{k=1}^{\infty}(A/r)^k)$ is the inverse of $r - A$. So $\sigma(A) \subset B_0(\|A\|)$ is bounded.

Let U be the set of invertible elements in $\mathcal{B}(\mathcal{H})$. Then U is open. Since the map $f : \mathbb{C} \to \mathcal{B}(\mathcal{H})$, $\lambda \mapsto \lambda - A$, is continuous and $\rho(A) = f^{-1}(U)$, we have $\rho(A)$ is open. So $\sigma(A)$ is closed.

The proof of Proposition 2.3.2 is completed. □

In the proof of Proposition 2.3.2, we consider a continuous map $f : \mathbb{C} \to \mathcal{B}(\mathcal{H})$. The norm of the bounded linear operator makes $\mathcal{B}(\mathcal{H})$ a Banach space, which is a topological space with topology induced by the norm. We need to say more here.

Differentiation

For $\lambda_0 \in \mathbb{C}$, we could define the derivative of f at λ_0 by

$$f'(\lambda_0) := \lim_{z \to 0} z^{-1}(f(\lambda_0 + z) - f(\lambda_0)) \tag{2.3.3}$$

if the limit exists with respect to the operator norm. Let Ω be an open subset of $\mathbb{C}$. The function f is called **differentiable** if for each $\lambda_0 \in \Omega$, the derivative of f at λ_0 exists. It is called differentiable on Ω if it is differentiable at each $\lambda_0 \in \Omega$.

Integral

Let $\Gamma = \{\Gamma(t) : t \in [0,1]\}$ be a rectifiable curve in Ω. Then $\int_\Gamma f(\lambda)d\lambda$ is defined as the limit in $\mathcal{B}(\mathcal{H})$ of sums of the form

$$\sum_j (\Gamma(t_j) - \Gamma(t_{j-1}))f(\Gamma(t_j)) \in \mathcal{B}(\mathcal{H}) \tag{2.3.4}$$

as we did in the course of Mathematical Analysis, where $\{t_0, \cdots, t_n\}$ is a partition of $[0,1]$. In the same way, we could show that if f is continuous, the limit exists.

An open subset $\Delta \subset \mathbb{C}$ is called a **Cauchy domain** if it is a disjoint union of a finite number of open connected sets $\Delta_1, \cdots, \Delta_r$, such that $\overline{\Delta_i} \cap \overline{\Delta_j} = \emptyset$ if $i \neq j$ and for each j the boundary of Δ_j consists of a finite number of non-intersecting closed rectifiable Jordan curves which are oriented in a way that Δ_j belongs to the inner domains of the curves. The oriented boundary of a bounded Cauchy domain in $\mathbb{C}$ is called a **Cauchy contour**. Usually we integrate a continuous function on a Cauchy contour. In fact, for any compact subset $K \subset \mathbb{C}$ and its any open neighborhood Ω, there exists a Cauchy domain Δ such that $K \subset \Delta \subset\subset \Omega$[3].

Analyticity

The function f is said to be **analytic** at $\lambda_0 \in \Omega$ if in some neighborhood U of λ_0 in Ω,

$$f(\lambda) = \sum_{n=0}^{\infty} (\lambda - \lambda_0) f_n, \quad \lambda \in U. \tag{2.3.5}$$

Here $f_0, f_1 \cdots \in \mathcal{B}(\mathcal{H})$ and the series (2.3.5) converges with respect to the operator norm. We say f is analytic on Ω if it is analytic at each $\lambda_0 \in \Omega$.

Theorem 2.3.3 (Cauchy Integral Formula)**.** *Assume that $f : \Omega \to \mathcal{B}(\mathcal{H})$ is analytic on Ω. Let Γ be a Cauchy contour such that Γ and its inner domain Δ are in Ω. Then for any $\lambda_0 \in \Delta$,*

$$f(\lambda_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\lambda)}{\lambda - \lambda_0} d\lambda. \tag{2.3.6}$$

In particular,

$$\frac{1}{2\pi i} \int_\Gamma f(\lambda) d\lambda = 0. \tag{2.3.7}$$

[3]cf. GTM 011, Proposition VIII.1.1 "Functions of one complex variables" by Conway.

Proof. Take an arbitrary continuous linear functional F on $\mathcal{B}(\mathcal{H})$. Then $F \circ f$ is an analytic function in the usual sense. From the usual Cauchy integral formula, we have

$$F[f(\lambda_0)] = \frac{1}{2\pi i}\int_\Gamma \frac{F[f(\lambda)]}{\lambda - \lambda_0} d\lambda. \tag{2.3.8}$$

On the other hand, from the definition of the integral (2.3.4), we have

$$F\left[\frac{1}{2\pi i}\int_\Gamma \frac{f(\lambda)}{\lambda - \lambda_0} d\lambda\right] = \frac{1}{2\pi i}\int_\Gamma \frac{F[f(\lambda)]}{\lambda - \lambda_0} d\lambda. \tag{2.3.9}$$

Since F is an arbitrary continuous linear functional on $\mathcal{B}(\mathcal{H})$, from the Hahn-Banach theorem[4], (2.3.8) and (2.3.9), we obtain (2.3.6).

If we replace $f(\lambda)$ in (2.3.9) by $f(\lambda)(\lambda - \lambda_0)$, we get (2.3.7).

The proof of Theorem 2.3.3 is completed. □

Theorem 2.3.4. *The function f is analytic on Ω if and only if f is differentiable on Ω.*

Proof. We only need to prove that differentiable implies analytic, which is a classical result in complex analysis in the usual sense.

Assume that f is differentiable on Ω. For any $\lambda_0 \in \Omega$, we could choose an oriented circle $\Gamma \subset \Omega$ with center at λ_0 and radius r such that its inner domain is also in Ω. Let F be an arbitrary continuous linear functional on $\mathcal{B}(\mathcal{H})$. Then the function $F \circ f$ is differentiable on Ω, and hence analytic on Ω. From the Cauchy integral formula,

$$F[f(\mu)] = \frac{1}{2\pi i}\int_\Gamma \frac{F[f(\lambda)]}{\lambda - \mu} d\lambda = F\left[\frac{1}{2\pi i}\int_\Gamma \frac{f(\lambda)}{\lambda - \mu} d\lambda\right], \quad |\mu - \lambda_0| < r. \tag{2.3.10}$$

So the Hahn-Banach theorem implies that

$$f(\mu) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\lambda)}{\lambda - \mu} d\lambda, \quad |\mu - \lambda_0| < r. \tag{2.3.11}$$

Since

$$\frac{1}{\lambda - \mu} = \frac{1}{(\lambda - \lambda_0)\left(1 - \frac{\mu - \lambda_0}{\lambda - \lambda_0}\right)} = \sum_{n=0}^{\infty} \frac{(\mu - \lambda_0)^n}{(\lambda - \lambda_0)^{n+1}}, \tag{2.3.12}$$

[4] Corollary 2.4.5 in "Functional Analysis I"

we have

$$f(\mu) = \sum_{n=0}^{\infty} (\mu - \lambda_0)^n \left(\frac{1}{2\pi i} \int_\Gamma \frac{f(\lambda)}{(\lambda - \lambda_0)^{n+1}} d\lambda \right). \tag{2.3.13}$$

So f is analytic on Ω.

The proof of Theorem 2.3.4 is completed. □

Lemma 2.3.5. *For $\lambda, \mu \in \rho(A)$, we have the resolvent equation*

$$(\lambda - A)^{-1} - (\mu - A)^{-1} = (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1}. \tag{2.3.14}$$

Proof. Multiply $(\lambda - A)$ on the left and $(\mu - A)$ on the right. □

Corollary 2.3.6. *The resolvent $(\lambda - A)^{-1}$ is analytic on $\lambda \in \rho(A)$.*

Proof. By Theorem 2.3.4 and Lemma 2.3.5, obvious. □

Definition 2.3.7. Let Ω be an open neighborhood of $\sigma(A)$. If $f : \Omega \to \mathbb{C}$ is analytic, for a Cauchy domain Δ with boundary Γ such that $\sigma(A) \subset \Delta \subset\subset \Omega$, we define $f(A) \in \mathcal{B}(\mathcal{H})$ by

$$f(A) := \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - A)^{-1} d\lambda. \tag{2.3.15}$$

From the Cauchy integration formula, $f(A)$ is independent of the choice of the Cauchy domain satisfying $\sigma(A) \subset \Delta \subset\subset \Omega$.

Let $\mathrm{Hol}(A)$ be the set of complex-valued functions which are analytic in a neighborhood of $\sigma(A)$. From the definition, it is easy to see that For any $f \in \mathrm{Hol}(A)$, $\alpha \in \mathbb{C}$,

$$(\alpha f)(A) = \alpha f(A). \tag{2.3.16}$$

Theorem 2.3.8 (Riesz Functional Calculus)**.** *(1) For any $f, g \in \mathrm{Hol}(A)$,*

$$(f + g)(A) = f(A) + g(A), \quad (f \cdot g)(A) = f(A)g(A). \tag{2.3.17}$$

(2) If $f \equiv 1$, then $f(A) = \mathrm{Id}$, i.e.,

$$\frac{1}{2\pi i} \int_\Gamma (\lambda - A)^{-1} d\lambda = \mathrm{Id}\,. \tag{2.3.18}$$

(3) If $f(z) = z^k$ for any $z \in \mathbb{C}$, $f(A) = A^k$, i.e.,

$$\frac{1}{2\pi i} \int_\Gamma \lambda^k (\lambda - A)^{-1} d\lambda = A^k. \tag{2.3.19}$$

(4) If $f, f_1, f_2, \cdots$ are analytic on Ω, and $f_n \to f$ uniformly on compact subsets of Ω, then $\|f_n(A) - f(A)\| \to 0$ as $n \to +\infty$.

(5) If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ has radius of convergence $R > \|A\|$, then $f \in$ $\mathrm{Hol}(A)$ and

$$f(A) = \sum_{k=0}^{\infty} a_k A^k. \tag{2.3.20}$$

Proof. The proof of the first equation of (2.3.17) is obvious. Let Ω be the open neighborhood of $\sigma(A)$ such that f and g are all analytic on Ω. Let Γ_1 and Γ_2 be Cauchy contours such that $\Gamma_1 = \partial\Delta_1$, $\Gamma_2 = \partial\Delta_2$ and $\sigma(A) \subset \Delta_1 \subset\subset \Delta_2 \subset\subset \Omega$. Then from Lemma 2.3.5,

$$\begin{aligned}
f(A)g(A) &= \left(\frac{1}{2\pi i}\int_{\Gamma_1} f(\lambda)(\lambda - A)^{-1}d\lambda\right)\left(\frac{1}{2\pi i}\int_{\Gamma_2} g(\mu)(\mu - A)^{-1}d\mu\right) \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} f(\lambda)g(\mu)(\lambda - A)^{-1}(\mu - A)^{-1}d\mu d\lambda \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} f(\lambda)g(\mu)\frac{(\lambda - A)^{-1}}{\mu - \lambda}d\mu d\lambda \\
&\quad - \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} f(\lambda)g(\mu)\frac{(\mu - A)^{-1}}{\mu - \lambda}d\mu d\lambda =: A - B.
\end{aligned} \tag{2.3.21}$$

So

$$\begin{aligned}
A &= \frac{1}{2\pi i}\int_{\Gamma_1} f(\lambda)(\lambda - A)^{-1}\left(\frac{1}{2\pi i}\int_{\Gamma_2}\frac{g(\mu)d\mu}{\mu - \lambda}\right)d\lambda \\
&= \frac{1}{2\pi i}\int_{\Gamma_1}(f\cdot g)(\lambda)(\lambda - A)^{-1}d\lambda = (f\cdot g)(A)
\end{aligned} \tag{2.3.22}$$

and

$$B = \frac{1}{2\pi i}\int_{\Gamma_2} g(\mu)(\mu - A)^{-1}\left(\frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(\lambda)d\lambda}{\mu - \lambda}\right)d\mu = 0. \tag{2.3.23}$$

(2) and (3). Let $f(z) = z^k$, $k \geq 0$. Let $\Gamma(t) = Re^{2\pi i t}$, $0 \leq t \leq 1$, $R > \|A\|$. Then

$$\begin{aligned}
f(A) &= \frac{1}{2\pi i}\int_{\Gamma}\lambda^k(\lambda - A)^{-1}d\lambda = \frac{1}{2\pi i}\int_{\Gamma}\lambda^{k-1}(1 - A/\lambda)^{-1}d\lambda \\
&= \frac{1}{2\pi i}\int_{\Gamma}\lambda^{k-1}\sum_{n=0}^{\infty}\frac{A^n}{\lambda^n}d\lambda.
\end{aligned} \tag{2.3.24}$$

Since the infinite series converges uniformly on Γ,

$$f(A) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\Gamma} \frac{1}{\lambda^{n-k+1}} d\lambda \right) A^n. \tag{2.3.25}$$

If $n \neq k$, $\int_\Gamma \frac{1}{\lambda^{n-k+1}} d\lambda = 0$. If $n = k$, $\int_\Gamma \frac{1}{\lambda^{n-k+1}} d\lambda = 2\pi i$. So $f(A) = A^k$.

(4). Let Δ be a Cauchy domain such that $\sigma(A) \subset \Delta \subset\subset \Omega$. Let $\Gamma = \partial\Delta = \cup_k \Gamma_k$, where Γ_k's are closed rectifiable Jordan curves. For k fixed,

$$\begin{aligned} &\left\| \int_{\Gamma_k} f_n(\lambda)(\lambda - A)^{-1} d\lambda - \int_{\Gamma_k} f(\lambda)(\lambda - A)^{-1} d\lambda \right\| \\ &\qquad = \left\| \int_0^1 (f_n(\Gamma_k(t)) - f(\Gamma_k(t)))(\Gamma_k(t) - A)^{-1} d\Gamma_k(t) \right\| \end{aligned} \tag{2.3.26}$$

Since $\|(\Gamma_k(t) - A)^{-1}\|$ is continuous on $t \in [0, 1]$ and bounded for any t, there exists $C > 0$ such that for any $t \in [0, 1]$, $\|(\Gamma_k(t) - A)^{-1}\| \leq C$. Let $|\Gamma_k|$ be the length of Γ_k. Then from (2.3.26), we have

$$\begin{aligned} &\left\| \int_{\Gamma_k} f_n(\lambda)(\lambda - A)^{-1} d\lambda - \int_{\Gamma_k} f(\lambda)(\lambda - A)^{-1} d\lambda \right\| \\ &\qquad \leq C|\Gamma_k| \max_{\lambda \in \Gamma_k} |f_n(\lambda) - f(\lambda)|. \end{aligned} \tag{2.3.27}$$

From the conditions, we have $\|f_n(A) - f(A)\| \to 0$ as $n \to +\infty$.

(5). Let $f_n(z) = \sum_{k=1}^n a_k z^k$. Then from (1)-(3), $f_n(A) = \sum_{k=1}^n a_k A^k$. Since $f(z) = \sum_{k=0}^\infty a_k z^k$ has radius of convergence $R > \|A\|$, $f \in \mathrm{Hol}(A)$ and $f_n(z) \to f(z)$ uniformly on compact subsets of $\{z : |z| < \mathbb{R}\}$. From (4), $f_n(A) \to f(A)$.

The proof of Theorem 2.3.8 is completed. □

Corollary 2.3.9. *Let $\sigma \subset \sigma(A)$ be a closed subset such that $\tau := \sigma(A) \backslash \sigma$ is also closed. Let Γ be a Cauchy contour such that σ is in the inner domain of Γ and τ is out of Γ. Then the operator*

$$P_\sigma := \frac{1}{2\pi i} \int_\Gamma (\lambda - A)^{-1} d\lambda \tag{2.3.28}$$

is a projection, i.e.,

$$P_\sigma^2 = P_\sigma. \tag{2.3.29}$$

Furthermore, we have

$$P_\sigma + P_\tau = \mathrm{Id}, \quad P_\sigma P_\tau = 0. \tag{2.3.30}$$

Proof. We could take a Cauchy domain $\Delta = \Delta_1 \cup \Delta_2$ such that $\sigma \subset \Delta_1$, $\tau \subset \Delta_2$ and $\overline{\Delta_1} \cap \overline{\Delta_2} = \emptyset$. Let $\Gamma_i = \partial\Delta_i$ for $i = 1, 2$. Take $f \equiv 1$ on Δ_1 and $\equiv 0$ on Δ_2. Then $f \in \mathrm{Hol}(A)$. So

$$f(A) = \frac{1}{2\pi i}\int_{\Gamma_1\cup\Gamma_2} f(\lambda)(\lambda - A)^{-1}d\lambda = \frac{1}{2\pi i}\int_{\Gamma_1} (\lambda - A)^{-1}d\lambda = P_\sigma. \quad (2.3.31)$$

Then from the second equality of (2.3.17), $P_\sigma^2 = P_\sigma$.

Take $g \equiv 1$ on Δ_2 and $\equiv 0$ on Δ_1. Then $g \in \mathrm{Hol}(A)$ and $g(A) = P_\tau$. Then from the first equality of (2.3.17), $P_\sigma + P_\tau = \mathrm{Id}$. In this case, $P_\sigma P_\tau = P_\sigma(1 - P_\sigma) = P_\sigma - P_\sigma = 0$.

The proof of Corollary 2.3.9 is completed. □

Theorem 2.3.10 (Spectral Mapping theorem)**.** *If $f \in \mathrm{Hol}(A)$, then*

$$\sigma(f(A)) = f(\sigma(A)). \quad (2.3.32)$$

Proof. If $\lambda_0 \in \sigma(A)$, let

$$g(\lambda) = \frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} \in \mathrm{Hol}(A). \quad (2.3.33)$$

If $f(\lambda_0) \notin \sigma(f(A))$, then $f(A) - f(\lambda_0)$ is invertible. From (2.2.31), $(\lambda_0 - A)^{-1} = g(A)(f(\lambda_0) - f(A))^{-1}$ is bounded, which is a contradiction of $\lambda_0 \in \sigma(A)$. So $f(\sigma(A)) \subset \sigma(f(A))$.

For the other direction, if $\mu \notin f(\sigma(A))$, then $g(\lambda) = (f(\lambda) - \mu)^{-1} \in \mathrm{Hol}(A)$. So $g(A)(f(A) - \mu) = \mathrm{Id}$. So $\mu \notin \sigma(f(A))$.

The proof of Theorem 2.3.10 is completed. □

Corollary 2.3.11. *Let $\sigma \subset \sigma(A)$ be a closed subset such that $\tau := \sigma(A)\backslash\sigma$ is also closed. Let $M = \mathrm{Im}P_\sigma$ and $L = \mathrm{Ker}\, P_\sigma$. Then $\mathcal{B}(\mathcal{H}) = M \oplus L$, the spaces M and L are A-invariant and*

$$\sigma(A|_M) = \sigma, \quad \sigma(A|_L) = \tau. \quad (2.3.34)$$

Proof. From (2.3.30), $L = \mathrm{Im}P_\tau$ and $\mathcal{B}(\mathcal{H}) = M \oplus L$. The spaces M and L are A-invariant. Take $f = z$ on Δ_1 and $\equiv 0$ on Δ_2. Then $f \in \mathrm{Hol}(A)$. So $A|_M = AP_\sigma = f(A)$. So (2.3.34) follows from Theorem 2.3.10.

The proof of Corollary 2.3.11 is completed. □

2.3.2 Functional Calculus for unbounded operators

In this subsection, we assume that A is an unbounded operator on $\mathcal{H}$ with domain $D(A)$. Remark that we don't need A is self-adjoint in this subsection.

Proposition 2.3.12. *The resolvent set $\rho(A)$ is open. If $\rho(A) \neq \emptyset$, then the resolvent $(\lambda - A)^{-1}$ is analytic on $\lambda \in \rho(A)$. Moreover, if $\lambda_0 \in \rho(A)$ and $|\lambda - \lambda_0| \leq \|(\lambda - A)^{-1}\|$, then $\lambda \in \rho(A)$ and*

$$(\lambda - A)^{-1} = \sum_{n=0}^{\infty} (-1)^n (\lambda - \lambda_0)^n (\lambda_0 - A)^{-(n+1)}. \tag{2.3.35}$$

Here this series converges in the operator norm. For $\lambda, \mu \in \rho(A)$, we have the resolvent equation

$$(\lambda - A)^{-1} - (\mu - A)^{-1} = (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1}. \tag{2.3.36}$$

Proof. The proof is the same as the bounded case. □

If A is unbounded, the spectrum $\sigma(A)$ is closed but unbounded. We need to compactify $\sigma(A)$ as follows.

Let $\mathbb{C}_\infty$ be the Riemann sphere, $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, endowed with the usual topology. The set $\mathbb{C}_\infty$ is a compact topological space and for any $\lambda \in \mathbb{C}$, the Möbius transformation

$$\eta_\lambda(z) := (\lambda - z)^{-1} \tag{2.3.37}$$

is a homeomorphism on $\mathbb{C}_\infty$.

Proposition 2.3.13. *Let A be an unbounded linear operator with non-empty resolvent set $\rho(A)$. Then for $\lambda \in \rho(A)$,*

$$\eta_\lambda(\sigma(A) \cup \{\infty\}) = \sigma((\lambda - A)^{-1}). \tag{2.3.38}$$

Proof. Note that $\eta_\lambda(\infty) = 0 \in \sigma((\lambda - A)^{-1})$, because $((\lambda - A)^{-1})^{-1} = \lambda - A$ is unbounded. For $z \neq \lambda$, $z \neq \infty$,

$$z - A = (\lambda - z)\left((\lambda - z)^{-1} - (\lambda - A)^{-1}\right)(\lambda - A). \tag{2.3.39}$$

So $z \in \sigma(A)$ if and only if $\eta_\lambda(z) = (\lambda - z)^{-1} \in \sigma((\lambda - A)^{-1})$.

The proof of Proposition 2.3.13 is completed. □

Since η_λ is a homeomorphism and $\sigma((\lambda - A)^{-1})$ is compact, we have $\sigma(A) \cup \{\infty\}$ is compact in $\mathbb{C}_\infty$.

Definition 2.3.14. Let Ω be an open neighborhood of $\sigma(A)$ in $\mathbb{C}$. Let Δ be a Cauchy domain such that $\sigma \subset \Delta \subset\subset \Omega$. Let $\Gamma = \partial\Delta$ be the Cauchy contour. In this case, some connected components are not closed. Let f be an analytic function on Ω. We assume that on any open connected component Γ_i of Γ,

$$|f(\Gamma_i(t))| \in \mathscr{S}(\mathbb{R}) \tag{2.3.40}$$

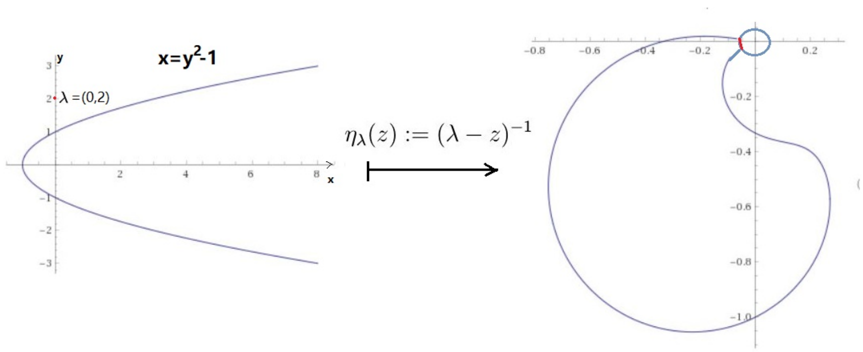
is a rapidly decreasing function (cf. (1.2.22)). We define

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda - A)^{-1} d\lambda, \tag{2.3.41}$$

Proposition 2.3.15. *The definition (2.3.41) is well-defined and does not depend on the Cauchy contour satisfying (2.3.40). Moreover, for $\lambda \notin \overline{\Delta}$, we have*

$$f(A) = \frac{1}{2\pi i} \int_{\eta_\lambda(\Gamma)} f \circ \eta_\lambda^{-1}(\mu)(\mu - (\lambda - A)^{-1})^{-1} d\mu. \tag{2.3.42}$$

Roughly speaking, we have $f(A) = (f \circ \eta_\lambda^{-1})((\lambda - A)^{-1})$.



Proof. We only prove (2.3.42). From (2.3.42), we could see that (2.3.41) is well-defined and does not depend on the Cauchy contour satisfying (2.3.40).

Let $B = (\lambda - A)^{-1}$, $z \in \rho(A)$, $z \neq \lambda$ and $\mu = (\lambda - z)^{-1}$. From (2.3.39),

$$(z - A)^{-1} = \mu B(\mu - B)^{-1} = \mu(\mu(\mu - B)^{-1} - \mathrm{Id}). \tag{2.3.43}$$

By taking $z = \eta_\lambda^{-1}(\mu) = \lambda - \mu^{-1}$, from (2.3.43), we have

$$\frac{1}{2\pi i} \int_{\Gamma} f(z)(z - A)^{-1} dz = \frac{1}{2\pi i} \int_{\eta_\lambda(\Gamma)} f(\eta_\lambda^{-1}(\mu))((\mu - B)^{-1} - \mu^{-1}) d\mu. \tag{2.3.44}$$

Let $\Gamma_\varepsilon = \partial(\eta_\lambda(\Delta) \backslash B_0(\varepsilon))$ for $\varepsilon > 0$ small enough. Then from (2.3.40), we have

$$\frac{1}{2\pi i} \int_{\eta_\lambda(\Gamma)} f(\eta_\lambda^{-1}(\mu))\mu^{-1} d\mu = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} f(\eta_\lambda^{-1}(\mu))\mu^{-1} d\mu. \tag{2.3.45}$$

Since $f \circ \eta_\lambda^{-1}$ is holomorphic on $\eta_\lambda(\Omega)$, by Cauchy integral formula, we have

$$\frac{1}{2\pi i}\int_{\Gamma_\varepsilon} f(\eta_\lambda^{-1}(\mu))\mu^{-1}d\mu = 0 \tag{2.3.46}$$

for any $\varepsilon > 0$. So

$$\frac{1}{2\pi i}\int_{\Gamma} f(z)(z-A)^{-1}dz = \frac{1}{2\pi i}\int_{\eta_\lambda(\Gamma)} f(\eta_\lambda^{-1}(\mu))(\mu - B)^{-1}d\mu. \tag{2.3.47}$$

The proof of Proposition 2.3.15 is completed. □

Remark 2.3.16. Remark that the assumption (2.3.40) is very strong. In fact, from the proof of Proposition 2.3.15, we only need the condition that the integral in (2.3.41) is well-defined.

From Theorems 2.3.8, 2.3.10 and Proposition 2.3.15, we obtain the following result.

Theorem 2.3.17. *For f, g be the analytic functions in Definition 2.3.14, we have*

$$(f+g)(A) = f(A) + g(A), \tag{2.3.48}$$

$$(\alpha f)(A) = \alpha f(A), \quad \forall \alpha \in \mathbb{C}, \tag{2.3.49}$$

$$(fg)(A) = f(A)g(A), \tag{2.3.50}$$

$$\sigma(f(A)) = f(\sigma(A)). \tag{2.3.51}$$

From Corollaries 2.3.9 and 2.3.11, we obtain the following result.

Theorem 2.3.18. *Let A be a closed operator with spectrum $\sigma(A) = \sigma \cup \tau$, where σ is contained in a bounded Cauchy domain Δ such that $\overline{\Delta} \cap \tau = \emptyset$. Let $\Gamma = \partial\Delta$. Then*

$$P_\sigma := \frac{1}{2\pi i}\int_\Gamma (\lambda - A)^{-1}d\lambda \tag{2.3.52}$$

is a projection. Let $M = \mathrm{Im} P_\sigma$ and $L = \mathrm{Ker}\, P_\sigma$. The spaces M and L are A-invariant and

$$\sigma(A|_M) = \sigma, \quad \sigma(A|_L) = \tau. \tag{2.3.53}$$

Furthermore, $M \subset D(A)$ and $A|_M$ is bounded.

2.3.3 Spectral decomposition for non-self-adjoint elliptic operator

Now we extend the spectral decomposition theorem (Theorem 2.2.47) to the non-self-adjoint case using the functional calculus.

Theorem 2.3.19. *Let $P : \mathscr{C}^\infty(M, F) \to \mathscr{C}^\infty(M, F)$ be an elliptic differential operator over a compact Riemannian manifold of order $m > 0$. Then for the spectrum $\sigma(P)$, there are two possibilities:*

(a) $\sigma(P) = \mathbb{C}$;

(b) $\sigma(P)$ is a discrete (maybe empty) subset of $\mathbb{C}$.

If (b) holds and $\lambda_0 \in \sigma(P)$, then there is a decomposition $L^2(M, F) = E_{\lambda_0} \oplus E'_{\lambda_0}$ such that the following conditions are satisfied:

(1) $E_{\lambda_0} \subset \mathscr{C}^\infty(M, F)$, $\dim E_{\lambda_0} < +\infty$, and E_{λ_0} is an invariant subspace of P such that there exists $N \in \mathbb{N}_+$ with $(P - \lambda_0)^N E_{\lambda_0} = 0$, i.e., the operator $P|_{E_{\lambda_0}}$ has only the eigenvalue λ_0 and is equal to the direct sum of Jordan cells of degree $\leq N$;

(2) E'_{λ_0} is a closed subspace of $L^2(M, F)$, invariant with respect to $\overline{P}$, i.e., $\overline{P}(D(\mathbf{H}^m(F) \cap E'_{\lambda_0})) \subset E'_{\lambda_0}$, and if we denote by $P'_{\lambda_0} := \overline{P}|_{E'_{\lambda_0}}$ as an unbounded operator in E'_{λ_0} with domain $\mathbf{H}^s(F) \cap E'_{\lambda_0}$, then $P'_{\lambda_0} - \lambda_0$ has a bounded inverse, i.e., $\lambda_0 \notin \sigma(P)|_{E'_{\lambda_0}}$.

Proof. Let $\lambda_0 \in \mathbb{C} \backslash \sigma(P)$, with loss of generality, assume that $\lambda_0 = 0$. So we have a bounded inverse $\overline{P}^{-1}$. Since $\overline{P}$ has positive order, by Rellich theorem, $\overline{P}^{-1}$ is a compact operator. Since

$$\overline{P} - \lambda = (\lambda^{-1} - \overline{P}^{-1})\lambda^{-1}\overline{P}, \tag{2.3.54}$$

$\lambda \in \sigma(P)$ if and only if $\lambda \neq 0$ and $\lambda^{-1} \in \sigma(\overline{P}^{-1})$. From the spectral theory of the compact operator, we see that $\sigma(P)$ is discrete.

Let $\sigma(P) \neq \mathbb{C}$, $\lambda_0 \in \sigma(A)$. Without loss of generality, we assume $\lambda_0 = 0$ again. Let Γ be the contour around 0 and not containing any other point of $\sigma(A)$. From Theorem 2.3.18, $P_0 = \frac{1}{2\pi i}\int_\Gamma (\lambda - A)^{-1} d\lambda$ is a projection, $E_{\lambda_0} = P_0(L^2(M, F))$, $E'_{\lambda_0} = (1 - P_0)(L^2(M, F))$ and E_{λ_0}, E'_{λ_0} are $\overline{P}$-invariant. From (2.3.53), $\lambda_0 \notin \sigma(P)|_{E'_{\lambda_0}}$. Since P is elliptic, $E_{\lambda_0} \subset \mathscr{C}^\infty(M, F)$ and is finite dimensional. From (2.3.53), $\sigma(P|_{E_{\lambda_0}}) = \lambda_0$. So $P|_{E_{\lambda_0}}$ is a linear transform on finite dimensional linear space E_{λ_0} with single eigenvalue λ_0. From the Jordan decomposition theorem in linear algebra, there exists $N \in \mathbb{N}_+$ with $(P - \lambda_0)^N E_{\lambda_0} = 0$.

The proof of Theorem 2.3.19 is completed. □

Remark that if $\mathrm{ind}(P) \neq 0$, $\sigma(P) = \mathbb{C}$. Because $\mathrm{ind}(P - \lambda) = \mathrm{ind}(P)$.

2.3.4 Complex powers of an elliptic operator

In this subsection, we introduce an important example of functional calculus of unbounded operator: complex powers of an elliptic operator, P^z, $z \in \mathbb{C}$, $\mathrm{Re} z < 0$.

Let $P : \mathscr{C}^\infty(M, F) \to \mathscr{C}^\infty(M, F)$ be an elliptic differential operator over a compact Riemannian manifold of order $m > 0$. We assume that $0 \notin \sigma(P)$. From Theorem 2.3.19, $\sigma(P)$ is a discrete set. We assume that there exists $\varepsilon > 0$ small enough and angle

$$\Lambda = \{\lambda \in \mathbb{C} : \pi - 2\varepsilon \leq \arg \lambda \leq \pi + 2\varepsilon\} \tag{2.3.55}$$

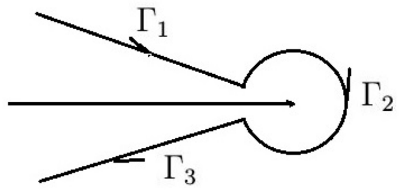
such that

$$\sigma(P) \cap \Lambda = \emptyset. \tag{2.3.56}$$

For $\lambda \in \mathbb{C}$, $z \in \mathbb{C}$, we have

$$\lambda^z = e^{z \ln \lambda} = e^{z \ln |\lambda| + iz \arg \lambda} = |\lambda|^z e^{iz \arg \lambda}. \tag{2.3.57}$$

Take $\rho > 0$ small enough that $B_0(2\rho) \cap \sigma(P) = \emptyset$. Consider the contour



where

$$\begin{aligned} \Gamma_1 : &\quad \lambda = re^{i(\pi-\varepsilon)}, \quad +\infty > r > \rho, \\ \Gamma_2 : &\quad \lambda = \rho e^{i\varphi}, \quad \pi - \varepsilon > \varphi > -\pi + \varepsilon, \\ \Gamma_3 : &\quad \lambda = re^{i(-\pi+\varepsilon)}, \quad \rho < r < +\infty. \end{aligned} \tag{2.3.58}$$

As in Definition 2.3.14, we define

$$P^z = \frac{1}{2\pi i} \int_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3} \lambda^z (\lambda - P)^{-1} d\lambda. \tag{2.3.59}$$

Then

$$\begin{aligned} P^z = \frac{1}{2\pi i} \int_{+\infty}^{\rho} & r^{\mathrm{Re} z} e^{-\mathrm{Im} z(\pi-\varepsilon)} e^{i(\mathrm{Im} z \ln r + \mathrm{Re} z(\pi-\varepsilon))} (re^{i(\pi-\varepsilon)} - P)^{-1} dr \\ & + \frac{1}{2\pi i} \int_{\pi-\varepsilon}^{-\pi+\varepsilon} \rho e^{-\varphi \mathrm{Im} z} e^{i\varphi \mathrm{Re} z} (\rho e^{i\varphi} - P)^{-1} i\rho e^{i\varphi} d\varphi \\ + \frac{1}{2\pi i} \int_{\rho}^{+\infty} & r^{\mathrm{Re} z} e^{-\mathrm{Im} z(-\pi+\varepsilon)} e^{i(\mathrm{Im} z \ln r + \mathrm{Re} z(-\pi+\varepsilon))} (re^{i(-\pi+\varepsilon)} - P)^{-1} dr \end{aligned} \tag{2.3.60}$$

Lemma 2.3.20. *For* $\lambda \in \Lambda$,

$$\|(P-\lambda)^{-1}\| \leq \frac{C}{|\lambda|}. \tag{2.3.61}$$

Proof. Add it in the future. □

From Lemma 2.3.20, Remark 2.3.16, (2.3.57) and (2.3.60), we see that P^z in (2.3.59) is well-defined for $\mathrm{Re} z < 0$ and bounded.

From the functional calculus Theorem 2.3.17, we have the following result.

Theorem 2.3.21. *(1) If* $\mathrm{Re} z < 0$, $\mathrm{Re} w < 0$, *then* $P^z P^w = P^{z+w}$.

(2) If $k \in \mathbb{Z}$, $k > 0$, *then* $P^{-k} = (P^{-1})^k$.

(3) If $\mathrm{Re} z < 0$, P^z *is holomophic on* $\mathrm{Re} z < 0$.